Unit 6F: Mathematical Induction

Today

- Continue: FOL axiomatic system
- Understand how to use 'mathematical induction'
- Take-home exercises
  - Example proofs (from lecture slides)
  - Informal proofs (everyday? theorems)
  - Proof by induction

Section 1

Deduction and Induction

- 'Deduction' [in general]: Argument in the direction of general to specific
  - E.g., From the fact that all living things will die, we conclude that "I will die."
- 'Induction' [in general]: Argument in the direction of specific to general
  - E.g., By looking at a lot of birds in a park, conclude that "all birds can fly."

Is induction always true?

Deduction in Logic

- Derivation of a theorem is also called 'deduction'
  - I.e., use of axioms and rules of inference (that are capable of proving all theorems) to obtain certain theorems

Induction in logic

- 'Inductive definition' (= 'recursive definition'): A definition of the entire set (or other objects) from a finite number of statements
  - E.g., define Newton's physics as a theory (collection of theorems) based on a small set of axioms
- 'Mathematical induction': A meta-proof of an infinite number of cases (i.e., general) from a finite number of formulas

Section Summary

- Deduction: general to specific
  - Always true
- Induction: specific to general
  - Not always true
  - We accept certain specific patterns, e.g., mathematical induction.

Section 2

Musical Chairs

- "The game of musical chairs eventually terminates."
  - Adopt the usual assumptions about the game
  - How to justify this statement?

'Mathematical Induction'

- Let **N** = $\{0, 1, 2, 3, ...\}$
- Let $P(n)$ be a predicate on $n \in$ **N**
- If both of the following holds:
  - $P(0)$
  - For every $n \in$ **N**, $P(n)$ implies $P(n + 1)$
- Then, $P(n)$ is true for every $n \in$ **N**
  - Note: A meta-theorem [only informally proven]

Variations: start from a non-0 value

Induction Example 1

- Theorem: The product of any 3 consecutive natural numbers is divisible by 6.
- Proof
  - Base case ($n = 0$): $0 \times 1 \times 2 = 0$ is divisible by 6.
  - Induction step: Suppose that the case holds for n, i.e., $n(n + 1)(n + 2)$ is divisible by 6. Then, at least one number is divisible by 2 and at least one (possibly the same) divisible by 3. Need to show the case for $n + 1$. Examine $(n + 1)(n + 2)(n + 3)$:
    - Case 1 (n is divisible by 6):
    - Case 2 (n is divisible by 2 but not by 3):
    - Case 3 (n is divisible by 3 but not by 2):
    - Case 4 (n is divisible neither by 3 nor by 2):

Induction Example 2

- Theorem: If $n + 1$ letters are put in n mailboxes, then some mailbox will contain (at least) two letters. [a special case of 'Pigeonhole Principle']
- Proof
  - Base case ($n = 1$): The single mailbox contains two letters.
  - Induction step: Suppose that the theorem holds for n. Show that it holds for $n + 1$ as well.
    - Case 1 (first mailbox has 2 letters): Done.
    - Case 2 (otherwise):

Induction and Recursion

- Both induction (mathematical/structural) and recursion are based on our informal understanding of the infinite whole from its finite components.
- Induction (mathematical/structural) depends on a theorem that a unique recursive function exists, which can be proven informally.

Section Summary

- Induction provides us with a powerful tool of dealing with infinite/complex objects.