Unit 7C: Counting

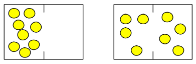
Today

- Apply 'counting' ideas to analyze the *irreversibility* of time
  - 'Entropy'
  - Probability
  - 'Permutation' and 'combination'
- Take-home exercises
  - *Alice in Wonderland*
  - Winning strategy
  - Additional exercises

Section 1

Irreversibility of Time

- Question: Although we can move in any direction in space, why are we limited to one direction with respect to time? cf. H.G. Wells
- One answer: The 'entropy' (degree of chaos) of a closed system always increases [Second Law of Thermodynamics]. Then, by statistical mechanics, the reversal is *extremely unlikely*.



'Entropy'

- If n events are likely to occur equally likely, its entropy H is $-\log_2(1/n)$
- Examples
  - Entropy of 2 equally likely event: 1
  - Entropy of 4 equally likely event: 2
  - Entropy of 8 equally likely event: 3
  - Entropy of 256 equally likely event: 8

unit of measurement?

Unit of Entropy

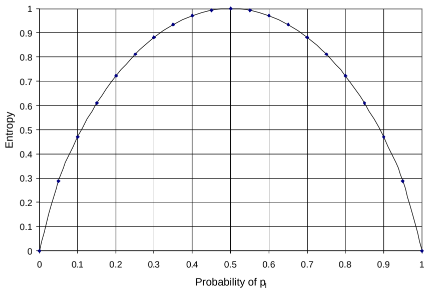
- Entropy with the base 2: measured in 'bits'
- One interpretation: The more bits we have (i.e., the more possibilities we can consider), the more chaotic (flexible) the system is.

Entropy in General

- The 'probability' p of an n equally-likely event is $1/n$. Then, $H = -\log_2 p$
- If there are n events with probability p_i each, $H = -\sum_{i=1}^{n} p_i \log_2 p_i$
- Examples

  cf. 0.5 vs. 0.5 ⇒ H = 1.0
  - Two events with probabilities 0.25 and 0.75: $H \approx 0.81$
  - Two events with probabilities 0.0 and 1.0: $H = 0.0$

Entropy of Two Events



Section Summary

- Entropy, based on the probability of each event, measures the degree of disorder, chaos, flexibility, randomness, informativeness, etc.
- More random ⇒ Higher entropy

Probability Basics

- Sample space, Ω: Set of all possible outcomes
- Event: $A \subseteq \Omega$
- Probability measure (a function) on Ω: $P(A)$ has a real value
  - $P(\Omega) = 1$
  - $A \subseteq \Omega \Rightarrow P(A) \geq 0$
  - $P(A \cup B) = P(A) + P(B)$ for disjoint A and B

function info?

logical specification of a structure (Ω, P, **R**)

Example

- Toss of fair coin twice
  - Sample space: W = **{*HH*, *HT*, *TH*, *TT*}**
  - Events: **{} = Æ, {*HH*}, {*HT*}, {*TH*}, {*TT*}, {*HH*, *HT*}, {*HH*, *TH*}, {*HH*, *TT*}, {*HT*, *TH*}, ..., {*HH*, *HT*, *TH*, *TT*}**
  - Probabilities: ***P*(Æ) = 0.0, *P*({*HH*}) = 0.25, ..., *P*({*HH*, *HT*}) = 0.5, ..., *P*({*HH*, *HT*, *TH*, *TT*}) = *P*(W) = 1.0**

‘Multiplication Principle’

- Suppose that two events occur *in sequence*, where the first event has n possibilities and the second, m. The number of possibilities for the sequence is $n \times m$.
- Examples
  - Throwing a die twice: $6 \times 6 = 36$
  - Choice of 5 appetizers and 9 main dishes: 45

‘Addition Principle’

- Suppose that two *disjoint* sets of objects, where the first set has n elements and the second, m. The number of objects in the two set is $n + m$.
- Example
  - Suppose that there are 4 players and 3 chairs (and no hybrids). Then, there are 7 objects.

Frequentist vs. Bayesian

- Frequentist view
  - Probability is objective.
  - It is a long-term average.
- Bayesian view
  - Probability is subjective.
  - Quantification of the observer’s uncertainty.

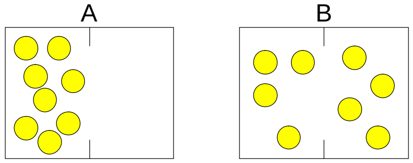
In general, there will be many* structures
that would satisfy a set of logical statements.

Section Summary

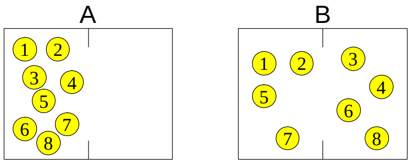
- Probability is a measure of likelihood for events in a sample space.
- We can now compute the probabilities of time reversal.

Section 3

Why is B more likely than A?



A Closer Look



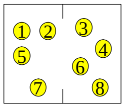
- A: Only one possibility
- B: Many possibilities of dividing objects

‘Permutation’

- The number of different *ordering* of r objects taken from n objects
  - Notation: $P(n, r)$
- Computation

  $P(n, r) = n \times (n - 1) \times \ldots \times (n - r + 1) = n!/(n - r)!$
- Example
  - Order 4 objects from 8:

    $P(8, 4)=1{,}680$



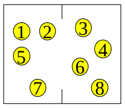
if numbered positions for objects

‘Combination’

- The number of different *choices* of r objects taken from n objects
  - Notation: $C(n, r)$
- Computation

  $C(n, r) = P(n, r) / r! = n!/(r!\,(n - r)!)$
- Example
  - Choose 4 objects from 8:

    $C(8, 4)=70$

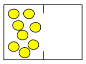


Possibilities for Case B

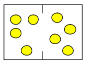
- 8-0 division: 1 (probability: $1/256 \approx 0.004$)
- 7-1 division: 8 (probability: 0.031)
- 6-2 division: 28 (probability: 0.110)
- 5-3 division: 56 (probability: 0.219)
- 4-4 division: 70 (probability: 0.273)
- 3-5 division: 56 (probability: 0.219)
- ...
- Total: 256 (probability: 1.000)

Section Summary

- The level of disorder and probability of each state can be computed by counting the possible occurrence for that state.
- Likelihood for the two cases:



1 possibility



256 possibilities

entropy?

Probability of time reversal: 1/256 ≈ 0.004

with atoms?