3-11.17

To find the eigenvalues of the given matrix, we write and solve the characteristic equation:

$$\begin{vmatrix} 5-\mu & 0 & 2 \\ 0 & 3-\mu & 0 \\ 2 & 0 & 5-\mu \end{vmatrix} = 0. \qquad \mu = 3, \quad (5-\mu)^2 = 4, \quad \text{so} \quad \mu = 7,\ 3,\ 3.$$

We note that $\mu = 3$ is an eigenvalue since, if $\mu = 3$, we have a column (or row) of zeros in the determinant. In that case, do not multiply the determinant out to obtain a cubic equation for μ, since you must then do unnecessary work to factor it! Instead, evaluate the determinant by a Laplace development using the row (or column) containing $(3-\mu)$; this gives $(3-\mu)[(5-\mu)^2 - 4] = 0$, or $\mu = 3$, $(5-\mu)^2 = 4$, as above.

To find the eigenvectors, we find (for each value of μ) a solution (x,y,z) of the matrix equation

$$\begin{pmatrix} 5-\mu & 0 & 2 \\ 0 & 3-\mu & 0 \\ 2 & 0 & 5-\mu \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

For $\mu = 7$, we have

$$\begin{cases} -2x \qquad +2z = 0, \\ \qquad -4y \qquad = 0, \\ 2x \qquad -2z = 0, \end{cases} \qquad \text{or} \qquad x = z, \quad y = 0.$$

Thus, an eigenvector corresponding to $\mu = 7$ is $(1,0,1)$ or $\vec{i} + \vec{k}$. The corresponding unit eigenvector is $\dfrac{\vec{i}+\vec{k}}{\sqrt{2}}$.

For $\mu = 3$, we find

$$\begin{cases} 2x + 2z = 0, \\ \qquad 0 = 0, \\ 2x + 2z = 0, \end{cases} \qquad \text{or} \qquad z = -x.$$

This is the equation of a plane; because $\mu = 3$ was a double eigenvalue, we find a whole plane of eigenvectors. We choose any two perpendicular eigenvectors in this plane, for example $(1,1,-1)$ and $(-1,2,1)$. (Note that both these eigenvectors have $z = -x$, and that the dot product of the two vectors is zero.) A simpler set of two perpendicular eigenvectors in the plane $z = -x$ is $(1,0,-1)$ and $(0,1,0)$; the corresponding unit eigenvectors are $\dfrac{\vec{i}-\vec{k}}{\sqrt{2}}$ and $\vec{j}$. Note that all the $\mu = 3$ eigenvectors are perpendicular to the eigenvector $\vec{i}+\vec{k}$ found above for $\mu = 7$.