Boas 6-10.5

Find $\iiint (\nabla\cdot\vec{F})d\tau$ over $x^2+y^2+z^2=25$

In order to use the divergence theorem, we need the unit vector $\vec{n}$ normal to the sphere. The vector $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$ is normal to the sphere; if we did not see this from the geometry, we could find it using the gradient: $\nabla(x^2 + y^2 + z^2) = 2\vec{r}$. Then the desired unit vector is $\vec{n} = \vec{r}/|\vec{r}| = (\vec{i}x + \vec{j}y + \vec{k}z)/5$ since $|\vec{r}| = r = (x^2 + y^2 + z^2)^{1/2} = 5$ on the surface of the sphere. We also want $\vec{F}\cdot\vec{n}$ when $r = 5$ since $\int\vec{F}\cdot\vec{n}d\sigma$ is an integral over the surface of the sphere. We find

$$\vec{F}\cdot\vec{n} = (x^2 + y^2 + z^2)(\vec{r}\cdot\vec{r}/5) = r^4/5 = 5^4/5 = 5^3.$$

Then, by the divergence theorem,

$$\int\nabla\cdot\vec{F}d\tau = \int\vec{F}\cdot\vec{n}d\sigma = 5^3\cdot(\text{area of sphere}) = 4\pi\cdot 5^5 = 12500\pi.$$