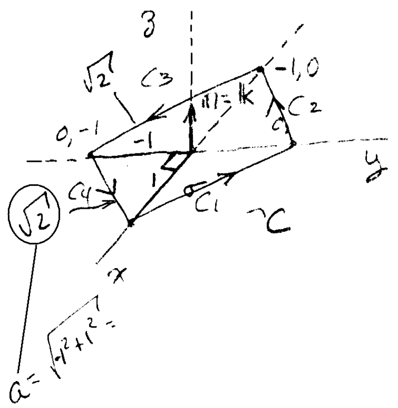
6-12.29

The line integral is equal to a surface integral over the area of the square by Stokes' theorem.

$$\mathbb{V} = x^2\,\mathbb{i} + 5x\,\mathbb{j}$$



$$(\nabla\times\mathbb{V})\cdot\mathbb{k} = \mathbb{k}\cdot\nabla\times\mathbb{V} = \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 5x & 0 \end{vmatrix}$$

$$(\nabla\times\mathbb{V})\cdot\mathbb{k} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 & 5x \end{vmatrix} = (5-0) = 5$$

Hence:

$$\oint_c \mathbb{V}\cdot d\mathbb{r} = \oint_\sigma (\nabla\times\mathbb{V})\cdot\mathbb{k}\,d\sigma$$

$$= \oint_\sigma 5\,d\sigma = 5\,(\text{area of square}) = 5\underbrace{(\sqrt{2})^2}_{A=2} = 10$$

$$\boxed{\oint_c \mathbb{V}\cdot d\mathbb{r} = 10}$$

$$\text{or } A = \left[\int_{x=-1}^{0}\int_{y=0}^{1+x} dy\,dx + \int_0^1\int_{y=0}^{1-x} dy\,dx + \int_1^0\int_{y=0}^{x-1} dy\,dx + \int_0^{-1}\int_{y=0}^{-x-1} dy\,dx\right]$$

$$= \left[\int_{-1}^{0}(1+x)dx + \int_0^1 (1-x)dx + \int_1^0 (x-1)dx + \int_0^{-1}(-x-1)dx\right] = \left[-(-1+\tfrac{1}{2}) + 1 - \tfrac{1}{2} - (\tfrac{1}{2}-1) + (-\tfrac{1}{2}+1)\right]$$

$$\left[1 - \tfrac{1}{2} + 1 - \tfrac{1}{2} - \tfrac{1}{2} + 1 - \tfrac{1}{2} + 1\right]$$

So $A = 2$ then $\oint_\sigma 5\,d\sigma = 5A = 5\cdot 2 = 10$

Also

$$\int \mathbb{V}\cdot d\mathbb{r} = \int x^2\,dx + 5x\,dy \qquad \text{where } d\mathbb{r} = \hat{\imath}\,dx + \hat{\jmath}\,dy$$

For C_1: $y = x-1$, $dy = dx$, $x = 0 \to 1$, $\int_{C_1} = \frac{17}{6}$

C_2: $y = -x+1$, $dy = -dx$, $x = 1 \to 0$, $\int_{C_2} = \frac{13}{6}$

C_3: $y = x+1$, $dy = dx$, $x = 0 \to -1$, $\int_{C_3} = \frac{13}{6}$

C_4: $y = -x-1$, $dy = -dx$, $x = -1 \to 0$, $\int_{C_4} = \frac{17}{6}$

$$\sum_i \int_{C_i} = \frac{17}{6} + \frac{13}{6} + \frac{13}{6} + \frac{17}{6} = \frac{60}{6} = 10$$